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Target space symmetries in topological theories I.

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Abstract: We study realization of the target space diffeomorphisms in the type C topological string. We found that the charges, which generate transformations of the boundary observables, form an algebra, which differs from that of bulk charges by the contribution of the bubbled disks. We discuss applications to noncommutative field theories.

Keywords: BRST Symmetry, Space-Time Symmetries, Anomalies in Field and String Theories.

Contents

1. Introduction

One of the great achievements of string theory is the construction of a quantum theory containing gravity. As such, the gauge symmetry of general relativity — space-time diffeomorphisms, or their α'-deformation must be present among the symmetries of string theory. Of course, the study of this symmetry or its deformation is obscured by a choice of a background metric $\langle g_{\mu\nu}\rangle \neq 0$ which leaves only a finite-dimensional group of isometries as explicit symmetry of the problem.

However, as we shall see below, there are string theories, which are formally related to a Seiberg-Witten $\alpha' \to 0$ limit (even in the case of ordinary bosonic string) of a physical string, which do not require target space metric at all. Instead, one deals with Poisson tensors, and sometimes with connections (but the dependence on the connection is in some sense trivial). The choice of a background Poisson tensor $\langle\theta^{\mu\nu}\rangle \neq 0$ is much less restrictive as far as the group of diffeomorphisms is concerned, for any hamiltonian vector field $V^\mu = \theta^{\mu\nu}\partial_\nu H$, where H is a function on the target space X, generates a symmetry of θ.

We are going to study these theories (they are called topological strings of type C) and will show that the closed strings enjoy the classical symmetry of Poisson diffeomorphisms, while the open strings exhibit a non-trivial deformation of this symmetry. The study of this deformed symmetry maybe a hint into what could be happening with the physical string symmetries in generic backgrounds.

The paper is organized as follows. In section 2, we describe the $\alpha' \to 0$ limit and the unconventional branch of (topological) string theories which emanate from this point. In section 3, we discover that in order to define the theory properly one is bound to utilize the techniques of BV quantization. We find a solution to BV master equation which enjoys target space covariance at the expense of introducing a connection on the tangent bundle to X. This new element appears upon careful examining of the properties of the auxilliary fields needed to ensure the proper gauge fixing. In section 4, we continue our study of the target space diffeomorphisms realized in the theory. We show that the closed string symmetries in general differ from those of open string. The origin of this anomaly is traced back to the phenomenon of “disk bubbling”, which is absent in the analogous quantum mechanical models. In section 5, we conclude by giving the possible applications of the discovered symmetry to the “covariant” noncommutative field theories.

Remarks on notations. Throughout the paper, we freely use the notions of topological field theories, like p-observables, ghost number, Witten's descend, Q-closedness etc, which are introduced in [1]. The target space coordinates are denoted by X^μ in section 2, and q^i in the rest of the paper. Poisson bi-vectors are denoted by $\theta^{\mu\nu}$ or π^{ij}.

Plans for the future. We plan to write an extended paper [2], which will contain the unified BV treatment of topological strings of types A, B, C, operator approach to topological (Hodge) quantum mechanics, more thorough treatment of the target space symmetries of topological field theories (beyond the dimensions ≤ 2), and applications to the recent formulation of superstring by Berkovits.

2. Seiberg-Witten limit and Poisson sigma model

2.1 Approaching from the physical side

Consider the action of bosonic string in the generic background of massless fields (without ghosts, see below):

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\ h^{1/2}\left[\left(g_{\mu\nu}(X)h^{ab} + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}(h)\Phi(X)\right], \quad (2.1)$$

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where Σ is the worldsheet Riemann surface with the metric h_{ab}, $h = \det h_{ab}$, and local coordinates $\sigma^a, a = 1,2$, $g_{\mu\nu}$ is the metric on the target space X, with local coordinates X^μ, and $B_{\mu\nu}$ is the two-form on X (more precisely, B is defined up to gauge transformations $B \to B + \Lambda$, where Λ is a closed two-from whose periods are in $8\pi^2\alpha'\mathbb{Z}$ so that globally on X, B needs not to be well-defined, and in fact couples to the word-sheet action via a Wess-Zumino term). For (2.1) to describe a conformal sigma model, the metric g, the B-field and the dilaton field Φ have to solve the beta-function equations:

$$\begin{aligned} R_{\mu\nu}(g) + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\lambda\omega}H_\nu^{\lambda\omega} &= O(\alpha') \\ \nabla^\omega\left(e^{-2\Phi}H_{\omega\mu\nu}\right) &= O(\alpha') \\ \frac{D-26}{6} + \alpha'\nabla_\omega\Phi\nabla^\omega\Phi - \frac{\alpha'}{2}\nabla^2\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} &= O(\alpha'^2)\,, \end{aligned} \tag{2.2}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of g, and $H = dB$. Let us re-write (2.1) in the first order form.[1] To this end introduce a one-form p_μ on Σ, with values in T^*X: $p_\mu = p_{\mu,a}\mathrm{d}\sigma^a$ and write an equivalent (after eliminating p) to (2.1) action:

$$S = \int_\Sigma i\ p_\mu \wedge \mathrm{d}X^\mu + \pi\alpha' G^{\mu\nu}(X)p_\mu \wedge \star p_\nu + \frac{1}{2}\theta^{\mu\nu}(X)p_\mu\wedge p_\nu + \text{dilatonic terms}\,, \tag{2.3}$$

where $\star$ is the two dimensional Hodge star operation on one-forms, which depends on $h^{1/2}h^{ab}$, $\star^2 = -1$, and

$$(g+B)^{-1} = G + \frac{\theta}{2\pi\alpha'}\,. \tag{2.4}$$

Imagine now taking the $\alpha' \to 0$ limit, while holding G and θ fixed (Seiberg-Witten limit [13]). It means that

$$\begin{aligned} g &\sim (2\pi\alpha')^2\frac{1}{\theta}G\frac{1}{\theta^t} \\ B &\sim 2\pi\alpha'\frac{1}{\theta}\,. \end{aligned} \tag{2.5}$$

From (2.2) we may now derive the $\alpha' \to 0$ limit of the beta-function conditions. We shall set $\Phi = 0$ for simplicity. Since $R_{\mu\nu}$ is invariant under the global rescaling of the metric the first term in the Einstein equation is $O(1)$, while the H^2 term scales as: ${\alpha'}^{-2}$, which forces $H = 0$. The next two equations are then automatically obeyed, as $\alpha' H^2 \sim {\alpha'}^{-1}$ dominates over $(D-26)/6 \sim \alpha'^0$ (at this point we assumed that θ is invertible).

Thus, we are approaching from the physical string side the "theory" with the action

$$S = \int_\Sigma i\ p_\mu \wedge dX^\mu + \frac{1}{2}\theta^{\mu\nu}p_\mu \wedge p_\nu\,, \tag{2.6}$$

where θ is such that $d\theta^{-1} = 0$. The last equation (for invertible θ) implies that θ is a Poisson tensor, i.e. if one defines a bracket on the functions on X by the formula

$$\{f,g\} = \theta^{\mu\nu}(X)\frac{\partial f}{\partial X^\mu}\frac{\partial g}{\partial X^\nu} \tag{2.7}$$

[1] Note that this is not the conventional passage to the hamiltonian framework, where one would have gotten a single component of p, thereby breaking two dimensional covariance.

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then it obeys Jacobi identity:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0\,. \tag{2.8}$$

2.2 Approaching from the topological side

Now imagine that we started (as in [3]) with (2.6) where θ is not necessarily an invertible Poisson tensor, i.e. $\theta^{\mu\nu}\partial_\nu\theta^{\lambda\omega}$ + cyclic permutations = 0. The theory with the action (2.6) has a symmetry descending from that of (2.1) — that of diffeomorphisms of X. It acts on p_μ as on the one-form on X, i.e. for the infinitesimal diffeomorphism,

$$\delta X^\mu = v^\mu(X), \;\; \delta p_\mu = -p_\nu \partial_\mu v^\nu\,. \tag{2.9}$$

Of course, the presence of g and B in (2.1) made them transform, thus making only a finite-dimensional subgroup of $\mathrm{Diff}(X)$ a symmetry, the rest acting on the space of backgrounds. Similarly, the presence of θ in (2.6) reduces $\mathrm{Diff}(X)$, but this time to an infinite-dimensional group $\mathrm{PDiff}(X,\theta)$ of Poisson diffeomorphisms. Any (2.9) with $v^\mu = \theta^{\mu\nu}\partial_\nu H$ for H a function on X generates a symmetry of (2.6).

In addition, the action (2.6) has a gauge symmetry [8]:

$$\begin{aligned} \delta_\varepsilon p_\mu &= d\varepsilon_\mu - \partial_\mu\theta^{\lambda\omega}\varepsilon_\lambda p_\omega \\ \delta_\varepsilon X^\mu &= \theta^{\mu\nu}\varepsilon_\nu\,. \end{aligned} \tag{2.10}$$

One can check that global symmetries are incompatible with the local symmetries. One can fix that by adding to $\delta_\varepsilon p_\mu$ the terms like $\Gamma^\lambda_{\mu\nu}\varepsilon_\lambda(dX^\nu - \theta^{\nu\omega}p_\omega)$, which depend on connection in tangent bundle to X and make everything covariant but then these transformations don't form a closed algebra (Q_{BRST} is not nilpotent). It is the Batalin-Vilkovisky (BV) formalism that saves the day, as we will see below.

Note. We do not claim that the bosonic string and the type C string are equivalent or continuously connected. They are clearly different ways of treating the ill-defined theory (2.6).

In the topological string we shall concentrate upon, the symmetry δ_ε is considered as a gauge symmetry and must be fixed. Also, to get back the amplitudes of the physical string one must fix the two-dimensional reparametrization invariance, which would add $b-c$ ghost system, fix Weyl invariance, while the gauge invariance (2.9) is broken explicitly by the coupling to the target space metric G. In the topological string the $b-c$ system is not needed — BV machinery will contain all the necessary ghosts and one can couple the system to the two dimensional topological gravity. At genus zero, which is what we shall study in this paper, this amounts to considering the integrals of the vertex operators over a compactification of the moduli space of points on the sphere (disk) up to the action of the group $\mathrm{SL}_2(\mathbb{C})$ ($\mathrm{SL}_2(\mathbb{R})$). The Feynman rules of [6, 8] automatically produce closed differential forms on these spaces.

3. Type C topological sigma model

3.1 A snapshot of BV formalism

Here we view the BV formalism (see, cf. [4]) as an integral I_{BV} of the BV differential form Ω_{BV} along the lagrangian submanifold $\mathcal{L}$ in the BV space:

$$I_{BV} = \int_{\mathcal{L}} \Omega_{BV} \,. \tag{3.1}$$

The BV space $\mathcal{M}$ is equipped with the canonical odd symplectic form ω_{BV}. One can choose local coordinates to identify $\mathcal{M}$ with $\Pi T^* N$ where N is some (super)manifold, where the symplectic form has a canonical form

$$\omega_{BV} = \delta Z_a^+ \wedge \delta Z^a \,, \tag{3.2}$$

where Z^a denotes the (super)coordinates on N and Z_a^+- corresponding coordinates on the cotangent fiber.

The submanifold $\mathcal{L}$ is lagrangian with respect to the canonical form ω_{BV} (in the physical literature its generating function is called the gauge fermion).

The BV differential form Ω_{BV} is constructed out of two ingredients [4]: the BV action S and the BV measure ν: $\Omega_{BV} = \left(\nu e^{-S}\right)$.

The action S must obey the so-called BV master equation:

$$\{S, S\}_{BV} := \omega_{BV}^{-1}(\partial_l S \wedge \partial_r S) = 0 \,. \tag{3.3}$$

One calls the coordinates Z^a the fields and Z_a^+ the anti-fields. Sometimes one distinguishes the classical part of N and the auxilliary fields used for gauge fixing. Also, the identification of the BV phase space with $\Pi T^* N$ is not unique and is not global in general, so the partition of all the fields involved on the fields and anti-fields is not unique.

The deformations of the action S that preserve (3.3) are (in the first order approximation) the functions Φ on $\mathcal{M}$ which are Q_{BV}-closed, where the differential Q_{BV} acts as $Q_{BV}\Phi = \{S, \Phi\}_{BV}$. The deformations which are Q_{BV}-exact are trivial in the sense that they could be removed by a symplectomorphism of $\mathcal{M}$ (one has to make sure that this symplectomorphism preserves ν to guarantee that the quantum theory is not sensitive to such a Q_{BV}-exact term).

3.2 Back to the $\int pdX$ theory

In this section we shall embed the action (2.6) into the BV framework. We start with the $\theta = 0$ case. We shall change the notations compared with the physical string case — the coordinates on X will be denoted mostly as q^i.

Consider the space $\mathcal{M}_{\Sigma,X}$ of maps of supermanifolds [5]:

$$\mathcal{M}_{\Sigma,X} = \mathrm{Maps}(\Pi T\Sigma, \Pi T^* X) \,.$$

If we choose on T^*X the coordinates (p_i, q^i), $i = 1, \ldots, \mathrm{dim} X$, with q^i being the coordinates on X, then $\varphi \in \mathcal{M}_{\Sigma,X}$ can be expressed via the following objects:

$$\varphi^* q^i = Q^i = q^i_{(0)} + q^i_{(1)} + q^i_{(2)} \,, \quad \varphi^8 p_i = Q^\dagger_i = p_{i(0)} + p_{i(1)} + p_{i(2)} \,, \tag{3.4}$$

where the component with a subscript (a) is a a-form on Σ, valued insome fiber bundle over Σ (the map from $\Pi T\Sigma$ is a collection of differential forms on Σ). In this expansion the pairs of field and antifields are just given by the pairs $(Q^i_{(a)}, Q^\dagger_{i(2-a)})$ for each value $0 \leq a \leq 2$.

In quantum field theory, $q^i = q^i_{(0)}$ and $p_i = p_{i(1)}$ are the classical fields present in the classical action, which is obtained from the BV action by setting all non-classical fields to zero. The component q^i_0 describes the ordinary map from Σ to X.

Now consider the effect of the change of coordinates

$$q^i \mapsto \tilde{q}^j(q)\,, \qquad p_i \mapsto \tilde{p}_i = p_j \frac{\partial q^j}{\partial \tilde{q}^i} \tag{3.5}$$

on $p_{i(a)}$ and $q^i_{(a)}$. Here $\frac{\partial q}{\partial \tilde{q}}$ is understood as the jacobian of the inverse change of coordinates.

From the definition of our expansions we readily compute:

$$\tilde{Q}^j = \tilde{q}^j(Q) = \tilde{q}^j(q_{(0)}) + \frac{\partial \tilde{q}^j}{\partial q^i} q^i_{(1)} + \frac{\partial \tilde{q}^j}{\partial q^i} q^i_{(2)} + \frac{1}{2}\frac{\partial^2 \tilde{q}^j}{\partial q^k \partial q^l} q^k_{(1)} q^l_{(1)} \tag{3.6}$$

and

$$\tilde{Q}^\dagger_j = Q^\dagger_i \frac{\partial Q^i}{\partial \tilde{Q}^j} \tag{3.7}$$

implies

$$\begin{aligned}
\tilde{p}_{j(0)} &= p_{i(0)} \frac{\partial q^i}{\partial \tilde{q}^j} \\
\tilde{p}_{j(1)} &= p_{i(1)} \frac{\partial q^i}{\partial \tilde{q}^j} - \frac{\partial q^m}{\partial \tilde{q}^l}\frac{\partial q^n}{\partial \tilde{q}^j}\frac{\partial^2 \tilde{q}^k}{\partial q^m \partial q^n} p_{k(0)} q^l_{(1)}
\end{aligned} \tag{3.8}$$

and we shall not need the explicit formula for $\tilde{p}_{j(2)}$ which is obtained by a straightforward tedious computation.

This transformations will be needed when adding to our systems of fields the BRST quartets that are needed for achieving the gauge fixing of the action (2.6). We will obtain a set of BRST transformations corresponding to a BV system of rank two. For such a system, the non linear antifield dependence forbids the use of the familiar Faddeev-Popov formula. It generally leads to a ghost and antighost dependence which is at least cubic. This makes the use of antifields unavoidable and will *a fortiori* justify the use of the BV formalism.

3.3 The W-deformation of the AKSZ action

Consider the action functional [5]:

$$S = \int_\Sigma Q^\dagger_i \mathrm{d}Q^i + \mathcal{W}(Q^\dagger, Q)\,, \tag{3.9}$$

where $\mathcal{W}(Q^\dagger, Q)$ is a (target-space scalar) function evaluated on the superfields Q and $Q^\dagger$. The integral in (3.9) picks out the two-form component $(Q^\dagger_i \mathrm{d}Q^i + \mathcal{W}(Q^\dagger, Q))|_{z\bar{z}}$. It is easy to verify that $\mathcal{W}(p_{(1)}, q_{(0)})$ gives (2.6). However the full content contains more information, and the superfield formalism with ghost unification simplifies tremendously all the formulae, as well as their geometrical interpretation.

S must obey the BV master equation:

$$\{S, S\} \equiv \sum_{i,a} \frac{\delta^r S}{\delta {p_i}^{(a)}} \frac{\delta^l S}{\delta {q^i}^{(2-a)}} - (-)^a (l \leftrightarrow r) = 0\,. \tag{3.10}$$

This implies that $\mathcal{W}$ obeys:

$$\frac{\delta \mathcal{W}}{\delta p_i} \frac{\delta \mathcal{W}}{\delta q^i} - (-)^a (l \leftrightarrow r) = 0\,. \tag{3.11}$$

In the example that we will study shortly in some detail

$$\mathcal{W} = \frac{1}{2} \pi^{ij}(Q) Q_i^+ Q_j^+ \tag{3.12}$$

and the condition (3.11) is equivalent to the statement that the bi-vector π is of Poisson type (2.8) (we changed the notation $\theta^{ij} \to \pi^{ij}$).

3.4 Quartets and Diff(X)

Let us now present the improvements needed in order to make the target space covariant gauge-fixing of the theory. Eventually, antighosts and Lagrange multipliers are needed for the gauge-fixing. We will generically denote them as χ and H, respectively. We thus consider the space $\mathcal{N}_{\Sigma,X}$ which is a fiber bundle over $\mathcal{M}_{\Sigma,X}$ with the fiber spanned by the quadruples $(\chi^i, H^i, \chi_i^+, H_i^+)$, where χ^i, H^i are fermionic and bosonic zero-forms on Σ with values in q^*TX (as usual q^* denotes the pull-back with respect to $q_{(0)}$).

As an example [2], if χ^i and H^i are fermionic and bosonic zero-forms on Σ, then $\chi_i^\dagger$ and $H_i^\dagger$ are respectively bosonic and fermionic two-forms with values in q^*T^*X.

Let us first show that the correct transformation law for the superfields $Q^\dagger, Q, \chi$, and so forth must be modified in the presence of anyone of the quartets $(\chi^i, H^i, \chi_i^+, H_i^+)$. We will find that the reparametrization invariance must be modified into:

$$\begin{aligned}
\tilde{Q}_l^\dagger &= Q_i^\dagger \frac{\partial Q^i}{\partial \tilde{Q}^l} - \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial q^n}{\partial \tilde{q}^l} \frac{\partial^2 \tilde{q}^i}{\partial q^m \partial q^n} \left(\chi_j^+ \chi^m + H_j^+ H^m \right) \\
\tilde{\chi}^i &= \chi^j \frac{\partial \tilde{q}^i}{\partial q^j}\,, \qquad \tilde{\chi}_i^+ = \chi_j^+ \frac{\partial q^j}{\partial \tilde{q}^i} \\
\tilde{H}^i &= H^j \frac{\partial \tilde{q}^i}{\partial q^j}, \quad \tilde{H}_i^+ = H_j^+ \frac{\partial q^j}{\partial \tilde{q}^i}\,.
\end{aligned} \tag{3.13}$$

In the first formula, one can use $Q, \tilde{Q}$ instead of $q, \tilde{q}$, since $\chi_j^+ \chi^n + H_j^+ H^n$ is always a two-form that automatically projects down to the 0-th component of the superfield Q.

The necessity of defining the coordinate transformations as in (3.13) is that we need a coordinate-invariant symplectic form on the space $\mathcal{M}$ for possibly defining a covariant path integral after the introduction of the fields χ and H. Indeed, $\mathcal{N}_{\Sigma,X}$ is endowed with the odd symplectic form, which is invariant under (3.13):

$$\Omega = \int_\Sigma \delta Q_i^\dagger \wedge \delta Q^i + \delta \chi^i \wedge \delta \chi_i^+ + \delta H_i^+ \wedge \delta H^i\,. \tag{3.14}$$

3.5 Naive BV action

As a first try, we assume that the BV action is equal to:

$$\mathcal{S}^{\text{naive}} = \int_{\Sigma} Q_i^{\dagger} \mathrm{d}Q^i + \mathcal{W}(Q^{\dagger}, Q) + \chi_i^{+} H^i \tag{3.15}$$

$\mathcal{S}^{\text{naive}}$ obeys the equation (3.10) if S does. It induces the following hamiltonian vector field action $\mathbf{s} = \{\mathcal{S}^{\text{naive}}, .\}$ on the space $\mathcal{N}_{\Sigma,X}$:

$$\begin{aligned}
\mathbf{s}Q_i^{\dagger} &= \mathrm{d}Q_i^{\dagger} - \frac{\partial \mathcal{W}}{\partial X^i}\,, \\
\mathbf{s}Q^i &= \mathrm{d}Q^i + \frac{\partial \mathcal{W}}{\partial Q_i^{\dagger}} \qquad \mathbf{s}\chi^i = H^i\,, \\
\mathbf{s}\chi_i^{+} &= H_i^{+} \mathbf{s}H^i = 0\,, \qquad \mathbf{s}H_i^{+} = 0
\end{aligned} \tag{3.16}$$

$\mathbf{s}$ is nilpotent due the fact that we have introduced the χ, H dependence without spoiling the BV master equation. The point is that it is necessary to define the transformation property of $Q^{\dagger}$ as in (3.13) in order that (3.14) be invariant. But then, the action $\mathcal{S}^{\text{naive}}$ (3.15), which satisfies the BV equation $\{\mathcal{S}^{\text{naive}}, \mathcal{S}^{\text{naive}}\} = 0$, is not invariant under (3.13).

3.6 Modified action

To solve this contradiction, we must modify $\mathcal{S}^{\text{naive}}$ into a new action. By trial and error, one finds an action $\mathcal{S}_{\gamma}$ that must explicitly depend on the choice of a connection $\gamma^i_{jk}(q^{(0)})$ on the tangent bundle TX to X. This modified action will be covariant with respect to (3.13) and still obey $\{\mathcal{S}_{\gamma}, \mathcal{S}_{\gamma}\} = 0$. The crucial subtlety is thus that one needs additional terms in order that the function $\mathcal{W}(Q^{\dagger}, Q)$ be coordinate-independent. This is a non-trivial requirement, given the intricate formula (3.13).

As a first attempt, one adds to $\mathcal{S}^{\text{naive}}$ the term:

$$\frac{\partial \mathcal{W}}{\partial Q_i^{\dagger}} \gamma^j_{ik} \left(\chi_j^{+} \chi^k + H_j^{+} H^k \right). \tag{3.17}$$

When one checks if this modified action obeys the master equation, one finds that it requires corrections that are non linear in γ. So, one needs higher order corrections to the action. Fortunately the procedure stops here with the following result:

$$\mathcal{S}_{\gamma} = \int_{\Sigma} Q_i^{\dagger} \mathrm{d}Q^i + \chi_i^{+} H^i + \mathcal{W}(Q^{\dagger}, Q) + \frac{\partial \mathcal{W}}{\partial Q_i^{\dagger}} \gamma^j_{ik} \left(\chi_j^{+} \chi^k + H_j^{+} H^k \right) + R(\gamma)^i_{jkl} \frac{\partial \mathcal{W}}{\partial Q_j^{\dagger}} \frac{\partial \mathcal{W}}{\partial Q_k^{\dagger}} \chi^l H_i^{+}\,. \tag{3.18}$$

In order to prove that $\mathcal{S}_{\gamma}$ obeys (3.10), we just need to prove the following:

$$\frac{\delta \mathcal{S}_{\gamma + t\alpha}}{\delta t}\bigg|_{t=0} = \{\mathcal{S}_{\gamma}, \mathcal{R}_{\alpha}\}\,, \tag{3.19}$$

where

$$\mathcal{R}_{\alpha} = \int \frac{\partial \mathcal{W}}{\partial Q_i^{\dagger}} \alpha^j_{ik} \chi^i H_j^{+}\,. \tag{3.20}$$

As a consequence, we have that for any value of the connection γ, $\{\mathcal{S}_\gamma, \mathcal{S}_\gamma\} = 0$. The proof of (3.19) is a simple computation. In particular, for $\gamma = 0$ the statement is trivial given (3.11). For $t \sim 0$ it is also simple since the last term in (3.18) can be neglected.

Given (3.19) the Poisson bracket $\{\mathcal{S}_\gamma, \mathcal{S}_\gamma\}$ is a solution to the first order differential equation in γ and hence vanishes in the light of the initial condition $\{\mathcal{S}_0, \mathcal{S}_0\} = 0$.

3.7 Boundary conditions

Here we specify the boundary conditions on the fields $(Q^i, Q_i^\dagger \chi^i, H^i, \chi_i^+, H_i^+)$, following [8]:

$$\begin{aligned} \star Q^i|_{\Pi T\partial\Sigma} &= 0\,, & Q_i^\dagger|_{\Pi T\partial\Sigma} &= 0\,, \\ \mathrm{d}\chi^i|_{\partial\Sigma} &= 0\,, & H^i|_{\partial\Sigma} &= 0\,, \\ \mathrm{d}\star\chi_i^+|_{\partial\Sigma} &= 0\,, & \star H_i^+|_{\partial\Sigma} &= 0\,. \end{aligned} \tag{3.21}$$

3.8 Diff(X) covariant gauge fixing of the C model

We now turn to the construction of an explicit gauge, that is, of a lagrangian submanifold $\mathcal{L} \subset \mathcal{N}_{\Sigma,X}$.

We first choose a function Ψ of half of the variables (the BV gauge function of ghost number -1), which we can call symbolically Z, which is going to be well-defined on $\mathcal{L}$ and such that

$$\delta\Psi = \delta^{-1}\omega_{BV}|_{\mathcal{L}}\,, \qquad \text{i.e.} \qquad Z^\dagger = \frac{\delta\Psi}{\delta Z}\,. \tag{3.22}$$

We find it convenient to perform a canonical transformation $(Q, Q^\dagger) \to (\rho, \xi)$:

$$\begin{aligned} \xi^i_{(0)} &= q^i_{(0)}\,, & \rho_{i(0)} &= p_{i(0)} \\ \xi^i_{(1)} &= q^i_{(1)}\,, & \rho_{i(1)} &= p_{i(1)} - \Gamma^j_{ik} q^k_{(1)} p_{j(0)} \\ \xi^i_{(2)} &= q^i_{(2)} + \frac{1}{2}\Gamma^i_{jk} q^j_{(1)} q^k_{(1)}\,, & \rho_{i(2)} &= p_{i(2)} - \frac{1}{2}\partial_i \Gamma^l_{jk} p_{l(0)} q^j_{(1)} q^k_{(1)} \end{aligned} \tag{3.23}$$

with the virtue that $\xi_{(0,1)}, \rho_{(0,1)}$ transform homogeneously under the coordinate transformations, unlike, say, $p_{i(1)}, p_{i(2)}, q^i_{(2)}$. Γ has to be a torsion-free connection on TX for (3.23) to be canonical. Denote

$$\nabla = \mathrm{d}\xi^l_{(0)} \nabla_l\,, \quad \nabla_l = \partial_l + \Gamma_l$$

and similarly for $\tilde{\nabla}$. We choose the following BV gauge function $\Psi(\rho_{i(0)}, \rho_{i(1)}, \xi^i_{(0)}, \chi^i, H^i)$ of the form:[2]

$$\Psi = \int_\Sigma \left(\mathrm{d}\chi^i + \tilde{\Gamma}^i_{jk}\chi^k \mathrm{d}\xi^j_{(0)}\right) \star \rho_{i(1)} = \int_\Sigma \chi^i \tilde{\nabla} \star \rho_{i(1)} \tag{3.24}$$

Ψ being H independent eliminates the Riemann tensor dependence from the action, since the BV constraint (3.22) gives that on $\mathcal{L}$:

$$\begin{aligned} H_i^+ &= 0\,, \qquad \chi_i^+ = -\tilde{\nabla} \star \rho_{i(1)} \\ \xi^i_{(1)} &= \star\tilde{\nabla}\chi^i\,, \qquad \xi^i_{(2)} = 0 \\ \rho_{i(2)} &= \partial_i \tilde{\Gamma}^l_{jk}\chi^k \mathrm{d}\xi^j_{(0)} \star \rho_{l(1)} - \mathrm{d}\left(\tilde{\Gamma}^k_{ij}\chi^j \star \rho_{k(1)}\right). \end{aligned} \tag{3.25}$$

[2]Notice that we have chosen that Ψ does not depend on H or χ^+. Otherwise we would have to introduce a metric G_{ij} on X. Our point is that we don't need to use metric, the connections $\Gamma, \Gamma', \ldots$ suffice. However, in [2] we will elaborate on the effect of a linear H dependence of Ψ which establishes a correspondence with A and B type models.

To simplify the notations in what follows we re-define:

$$
\begin{aligned}
q^i &= \xi^i_{(0)} \\
p_i &= \rho_{i(1)} \\
\theta_i &= \rho_{i(0)} \,.
\end{aligned}
\tag{3.26}
$$

As before, we take:

$$
\mathcal{W}(Q, Q^+) = \frac{1}{2}\pi^{ij}(Q) Q^+_i Q^+_j \tag{3.27}
$$

with π^{ij} being a Poisson bi-vector, i.e. bi-vector obeying (2.8). On $\mathcal{L}$ the original fields-antifields are:

$$
\begin{aligned}
Q^i &= q^i + \star\tilde{\nabla}\chi^i - \frac{1}{2}\Gamma^i_{jk} \star \tilde{\nabla}\chi^j \star \tilde{\nabla}\chi^k \\
Q^\dagger_i &= \theta_i + p_i + \Gamma^j_{ik}\theta_j \star \tilde{\nabla}\chi^k + \frac{1}{2}\partial_i\Gamma^l_{jk}\theta_l \star \tilde{\nabla}\chi^j \star \tilde{\nabla}\chi^k + \tilde{\Gamma}^l_{ik}\left(p_l \star \tilde{\nabla}\chi^k + \chi^k\tilde{\nabla} \star p_l\right) + \\
&\quad + \tilde{R}^l_{ijk}\chi^k dq^j \star p_l \,.
\end{aligned}
\tag{3.28}
$$

Let us make the final adjustment of the notations: introduce:

$$
h^i = H^i + \chi^k \left(\tilde{\Gamma} - \gamma\right)^i_{jk} \pi^{jl}\theta_l \,. \tag{3.29}
$$

Note that the formula for Q^i can be intepreted as an equation for a formal geodesic in X with respect to the connection Γ, which starts at the point q along the formal tangent vector $\star\tilde{\nabla}\chi$. It is plausible that the similar relations will hold in the higher-dimensional analogues of the type C sigma models. The restriction of the action functional $\mathcal{S}$ on $\mathcal{L}$ (= the gauge fixed action) is given by (note that γ has totally disappeared from the final lagrangian):

$$
\begin{aligned}
\mathcal{S}_{\rm gf} = &\int_\Sigma p_i(\mathrm{d}q^i + \star\tilde{\nabla}h^i) + \theta_i\nabla \star \tilde{\nabla}\chi^i + \frac{1}{2}\pi^{ij}p_ip_j + \star\tilde{\nabla}\chi^k\tilde{\nabla}_k\pi^{ij}p_i\theta_j + \\
&+ \tilde{R}^l_{ijk}\pi^{im}\chi^k\theta_m dq^j \star p_l - \theta_i\theta_j\tilde{\nabla}\chi^a\tilde{\nabla}\chi^b\left(\frac{1}{2}\nabla^2_{ab}\pi^{ij} - R^i_{cab}\pi^{cj}\right) .
\end{aligned}
\tag{3.30}
$$

The beauty of our action (3.18) seems lost when components are made explicit, and the built-in invariances seem awkward under this form, yet the coordinate covariance is manifest. Also, (3.30) shows that we have a system of rank two, due to the term $\partial^2\pi\tilde{\nabla}\chi\tilde{\nabla}\chi\theta\theta$ which is quadratic in the antifields $q^i_{(1)}$. Since the Ψ is $p_{(1)}$-dependent, this term gives a non-trivial quartic ghost dependence of the gauge-fixed action. The later cannot be generated by some kind of Faddeev-Popov determinant, which is the justification for the whole BV machinery. Such terms are different in nature from those that one obtains in the A model through the curvature dependent terms. The C model is thus quite different of the A-model, for which the gauge function is independent on p, which implies that it can be analyzed in the usual BRST formalism as a first rank system.

3.9 Covariant deformation quantization

The final action (3.30) has the virtue of being a sigma model action, i.e. it is well-defined in terms of the geometrical data, i.e. the maps of the worldsheet into the target space endowed with the connections $\Gamma, \tilde{\Gamma}$ on its tangent bundle, as well as the Poisson bi-vector field π. It is not obvious that the action (3.30) defines a (super)conformal field theory for non-flat $\Gamma, \tilde{\Gamma}$, however, the RG flow, if any, should not affect the correlation functions of the Q-invariant observables. Thus one expects that the correlation function:

$$\langle f_1(q(0))\, f_2(q(1))\rangle_{q(\infty)=q} =: f_1 \star f_2(q) \tag{3.31}$$

defines an associative star-product. This star-product will, nevertheless, depend on the connection $\tilde{\Gamma}$, even though all $\tilde{\Gamma}$ dependence comes through Q-exact terms. Let us see how this can happen. We have no other way of treating the theory defined by (3.30) but by perturbation expansion. In order to generate the perurbation series we choose a classical solution $q(z,\bar{z}) = q$, $p = 0$, $\chi = 0$, $\theta = 0$, $\ldots$, and expand around it. We should also keep track of the covariance properties of our expansion. As is standard in the sigma model techniques, [14] it is convenient to use the locally geodesic coordinates, which identify the vicinity of the point q in the target space with the vicinity of zero in the tangent space to the manifold at this point:

$$\begin{aligned} q^i(z,\bar{z}) &= q^i + \mathbf{y}^i - \frac{1}{2}\tilde{\Gamma}^i_{jk}(q)\mathbf{y}^j\mathbf{y}^k + \cdots\,, \qquad \mathbf{y}^i = \mathbf{y}^i(z,\bar{z}) \\ p_i(z,\bar{z}) &= \mathbf{p}_i + \tilde{\Gamma}^j_{ik}(q)\mathbf{y}^k\mathbf{p}_j + \cdots\,, \\ h^i(z,\bar{z}) &= \mathbf{h}^i - \tilde{\Gamma}^i_{jk}(q)\mathbf{h}^j\mathbf{y}^k + \cdots \end{aligned} \tag{3.32}$$

and similarly for fermions. At this point we assume that $\tilde{\Gamma}$ is also torsion-free. We can then set $\Gamma = \tilde{\Gamma}$ for simplicity. Then the action (3.30) becomes an infinite expansion in $\mathbf{y}$'s, with each vertex being constructed out of covariant expressions, like the curvatures, their covariant derivatives, the covariant derivatives of π and so on, all at the point q. Similarly, the boundary observables $f(q(0))$ etc. become the expansions in $\mathbf{y}$ whose coefficients are nothing but the iterated (and symmetrized) covariand derivatives of f at the point q. The action now has the form:

$$S = S_0 + I \ldots\,,$$

where

$$S_0 = \int \mathbf{p}_i d\mathbf{y}^i + \mathbf{p}_i \star d\mathbf{h}^i + \text{fermions}$$

and I is the rest (containting π). We now expand e^{-S} in I. The important feature of the interaction density $\mathcal{L}$, $I = \int \mathcal{L}$ is that it obeys the descend relations $\{Q, \mathcal{L}\} = d\mathcal{O}^{(1)}$ for some 1-forms $\mathcal{O}_k$.

Let us vary the correlation function (3.31) with respect to $\tilde{\Gamma}$. It brings down the extra terms of the form $\int\{Q, \delta R\}, R = \nabla_i \delta\tilde{\Gamma}^j_{kl}\mathbf{y}^i d\mathbf{y}^k \star \mathbf{p}_j \chi^l + \cdots\}$. Normally we would use the fact that Q is a symmetry of the theory to pull Q off this term to make it act on the rest of the correlation function. This operation will convert the integrals

$$\frac{1}{k!}\int \langle \{Q, R\}(z)\mathcal{L}(w_1)\ldots\mathcal{L}(w_k)\rangle$$

into

$$\frac{1}{k!}\int\left\langle R(z)\sum_{i=1}^{k} d\mathcal{O}(w_i)\prod_{j\neq i}\mathcal{L}(w_j)\right\rangle .$$

The total derivative $d\mathcal{O}$ would make the integrated correlation function vanish if there were no boundaries in the integration domain. There are two kinds of boundaries: $w_i \to w_j$ and $w_i \to$ the boundary of the worldsheet. We argue in [2] that these boundary contributions are non-zero, thus providing the mechanism for non-decoupling of the Q-exact terms, needed for covariantization.

In the next subsections we shall not keep these Γ-dependent terms, instead we shall analyze the currents and the charges generating the target space diffeomorphisms and will see the origin of the non-covariance of the original star product from a slightly different yet related perspective.

4. The symmetries of bulk and boundary theories

4.1 Generalities on quantum symmetries in the bulk

Recall that, in ordinary theories, the quantum global symmetry is generated by a current J_V that is conserved

$$dJ_V = 0 \tag{4.1}$$

(inside correlation functions). Moreover the quantum action of the global symmetry on the observable $\mathcal{O}(x)$ inserted at point x is given by the insertion of the expression

$$\int_{S(x,r)} J_V$$

under the correlator:

$$\lim_{r\to 0}\left\langle \ldots \int_{S(x,r)} J_V\mathcal{O}(x)\ldots\right\rangle = \langle \ldots \delta_V\mathcal{O}(x)\ldots\rangle . \tag{4.2}$$

Here $S(x,r)$ is the set of points y such that the distance between x and y equals r, and $r\to 0$.

If the BV action and the measure are invariant under a global symmetry, but the gauge fixing, that is a choice of lagrangian submanifold $\mathcal{L}$ is not invariant, then the current is conserved up to the Q_{BV} exact terms, i.e.

$$dJ_V = Q_{BV}(J_{V,(2)}) . \tag{4.3}$$

This current is Q_{BV} — closed, up to a total derivative, i.e.:

$$Q_{BV}(J_V) = dJ_{V,(0)} . \tag{4.4}$$

Note, that the above equation means that current is the density of a topological observable that is related to 2 and 0–observables by Witten's descent equation [1]. The currents in the bulk form the bulk algebra that is defined as follows. Let us insert the 0-observable

$J_{V_2,(0)}$ at the point x in the bulk and integrate the current J_{V_1} along a small circle around x. This produces the zero observable at x, which we will denote as $J_{[V_1,V_2]_{bulk},(0)}(x)$:

$$\lim_{r\to 0}\left\langle \int_{z\in S(x,r)} J_{V_1}(z)J_{V_2,(0)}(x)\right\rangle = \langle \dots J_{[V_1,V_2]_{bulk},(0)}(x)\dots\rangle\,. \tag{4.5}$$

4.2 Generalities on 2d topological theories on surfaces with boundaries

If a topological theory is defined on a surface with boundaries, we must choose boundary conditions that preserve Q_{BV}. In particular, the current J_{BV} that generates the Q_{BV} symmetry must vanish (within correlation functions) on the boundary. Similarly, we have an additional condition for the current J_V to generate a symmetry: it must vanish when restricted to the boundary within a correlation function:

$$\langle \dots J_V|_{\partial\Sigma}\dots\rangle = 0\,. \tag{4.6}$$

The boundary 0-observables $\mathcal{O}_f(x)$ are those that do not change the vanishing of the current J_{BV} when they are put on the boundary $x\to\partial\Sigma$. Here f is some label on the space of 0-observables. In the type C topological string f stands for a function on the target space.

Since the energy-momentum tensor in topological theory is Q_{BV}-exact, the correlators of local observables on the boundary are not changing when we smoothly move the insertion point, without colliding with another insertion point.

By moving the insertion points together we find that the correlators of local observables on the boundary are governed by the so call $*$-product on local boundary observables:

$$\langle \dots \mathcal{O}_{f_1}(x_1)\mathcal{O}_{f_2}(x_2)\rangle = \lim_{x_1\to x_2}\langle \dots \mathcal{O}_{f_1}(x_1)\mathcal{O}_{f_2}(x_2)\rangle = \langle \dots \mathcal{O}_{f_1 * f_2}(x_2)\rangle \tag{4.7}$$

if there are no other observables between x_1 and x_2. Here x_1, x_2 are two points on he boundary $\partial\Sigma$.

Now, we can define the action U_V of symmetry on the 0-observable on the boundary $\mathcal{O}_f(x)$ as follows:

$$\langle \dots (\mathcal{O})_{U_V(f)}(x)\rangle = \lim_{r\to 0}\left\langle \dots \int_{S(x,r)} J_V\mathcal{O}_f(x)\right\rangle\,, \tag{4.8}$$

where $\dots$ stands for insertions of other Q_{BV}-closed observables whose support (range of integration) does not intersect the arc $S(x,r)$ of bulk points y situated at the distance r from x. Notice that it contains quantum corrections as compared to the naive classical symmetry action.

We will be interested in the computation of the Lie algebra of actions on the boundary observables whose structure constants C_{ij}^k are defined as

$$U_{V_i}(U_{V_j}(f)) - U_{V_j}(U_{V_i}(f)) = C_{ij}^k U_{V_k}(f)\,. \tag{4.9}$$

As we will see below this algebra is different from the bulk algebra defined in (4.5).

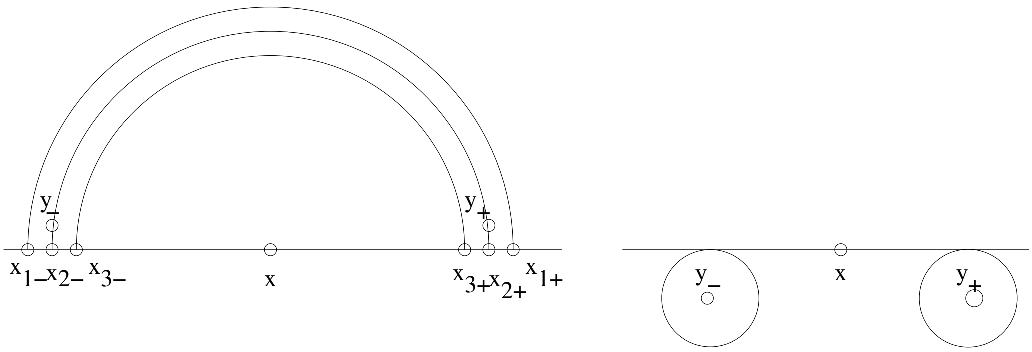

Figure 1: Setup for the commutator calculation. Bubbled discs.

4.3 Disk bubbling and deformation of the boundary algebra

We are going to study here the correlator that determines the commutator (4.9) in conformal topological theory :

$$\langle \ldots \mathcal{O}_{[U_{V_1},U_{V_2}](f)}(x)\rangle = \left\langle \ldots \int_{S(x,r_1)} J_{V_1} \int_{S(x,r_2)} J_{V_2}\mathcal{O}_f(x)\ldots \right\rangle - \\ - \left\langle \ldots \int_{S(x,r_3)} J_{V_1} \int_{S(x,r_2)} J_{V_2}\mathcal{O}_f(x)\ldots \right\rangle , \qquad (4.10)$$

where $r_1 > r_2 > r_3$. Let us denote the beginning of the i-th arc and the end of the i-th arc as $x_{-,i}$ and $x_{+,i}$ respectively.

It is clear that the r.h.s. of (4.10) is independent on the exact value of x_i - the only thing that matters is that the intervals $[x_{i,-}, x_{i,+}]$ are arranged as follows:

$$[x_{3,-}, x_{3,+}] \subset [x_{2,-}, x_{2,+}] \subset [x_{1,-}, x_{1,+}] \,. \qquad (4.11)$$

We will take the arcs close to each other, i.e.

$$r_1 - r_3 \ll r_3 \,. \qquad (4.12)$$

The commutator appears as a result of deformation of the arc $S(x, r_1)$ into the arc $S(x, r_3)$. The easiest way to achieve this deformation would be to start with the replacement of the arc $S(x, r_2)$ by the arc $S'(x, r_2)$ that connects the points $y_\pm$ obtained from $x_{2,\pm}$ by very small shifts inside the disk, namely

$$|y_\pm - x_{2,\pm}| \ll r_1 - r_3 \ll r_3 \,. \qquad (4.13)$$

Then, using the vanishing of current J_{V_1} on the boundary and applying Stokes theorem, we can rewrite the commutator as a sum of two terms:

$$\left\langle \ldots \mathcal{O}_{[U_{V_1},U_{V_2}](f)}(x)\ldots \right\rangle = \left\langle \ldots \int_{y\in\Gamma_{y_-,y_+}} \int_{z\in\Gamma_y} J_{V_1}(z)J_{V_2}(y)\mathcal{O}_f(x)\ldots \right\rangle + \qquad (4.14) \\ + \left\langle \ldots \int_{y\in\Gamma_{y_-,y_+}} J_{V_2}(y) \int_{z\in SA} Q_{BV}(J_{V_1,(2)}(z))\mathcal{O}_f(x)\ldots \right\rangle ,$$

where Γ_{y_-,y_+} is an arc that connects points y_- and y_+, Γ_y is a small loop around point y, and SA is the semi-annulus, whose boundary consists of the two intervals $[x_{1,-}, x_{3,-}]$, $[x_{1,+}, x_{3,+}]$ and the two arcs.

The first term in (4.14) is just the contribution from the OPE of the currents, i.e., is given by the bulk algebra (4.5).

The second term in (4.14) is more interesting. Taking Q_{BV} from $J_{V_1,(2)}$ and applying it to J_{V_2} we will get total derivative $dJ_{V_2,(0)}$, see (4.4). We can integrate this total derivative to get the following expression for the second term:

$$\left\langle \dots \int_{y\in\Gamma_{y_-,y_+}} J_{V_2}(y) \int_{z\in SA} Q_{BV}(J_{V_1,(2)}(z))\mathcal{O}_f(x)\dots \right\rangle = \\ = \left\langle \dots \int_{z\in SA} (J_{V_2,(0)}(y_+) - J_{V_2,(0)}(y_-))J_{V_1,(2)}(z)\mathcal{O}_f(x)\dots \right\rangle . \qquad (4.15)$$

Now an interesting thing happens. In massive theory the contribution of SA would be negligible since its area is small. In a conformal theory the notion of the absolute area makes no sense. In order to see what really contributes, we make a conformal transformation that maps the area around the points $y_\pm$ to the disks with centers $y_\pm$ which are bubbled out. The rest of the semi-annulus SA is mapped into a figure connecting these bubbled out disks — one can show that the contribution of the rest of the SA could be neglected.

The integral of $J_{V_1,(2)}$ over the bubbled disk BD, with the operator $J_{V_2,(0)}$ at its center, can be replaced by a 0-observable $\mathcal{O}_{F_{BD}(V_1,V_2)}$ placed at the point where bubbled disk joins the rest of the surface:

$$\langle \dots \mathcal{O}_{F_{BD}(V_1,V_2)} \dots \rangle = \left\langle \int_{z\in BD} J_{V_2,(0)}(y)J_{V_1,(2)}(z) \right\rangle . \qquad (4.16)$$

Thus, we get the following expression for the commutator of the U_V's:

$$[U_{V_1}, U_{V_2}](f) = U_{[V_1,V_2]_{bulk}}(f) + (f * F_{BD}(V_1,V_2) - F_{BD}(V_1,V_2) * f) . \qquad (4.17)$$

It is the second term which only occurs due to the presence of the boundary and it makes the algebra of boundary symmetries *different* from that in the bulk.

4.4 Example: Diff-symmetry of the $*$-product

Now we will show how general considerations above about the action of the symmetries on the boundary 0-observables work in the C model. The boundary 0-observables in the model are the functions on X (one can also consider differential forms of higher degree, but this does not give anything new), $\mathcal{O}_f(z) = f(q(z))$. The three-point function on the disk defines the $*$-product:

$$\langle \mathcal{O}_{f_1}(0)\mathcal{O}_{f_2}(1)\mathcal{O}_{f_3}(\infty)\rangle = \int_X \nu\ f_1 * f_2 * f_3 \, , \qquad (4.18)$$

where ν is some top degree form on X (the descendant of BV measure ν). It is convenient to work on the upper half-plane instead of the disk and to replace $f_3\nu$ by a $\delta(q(\infty) - x)$, so that

$$\langle \mathcal{O}_{f_1}(0)\mathcal{O}_{f_2}(1)\rangle_x = f_1 * f_2(x) = f_1f_2(x) + \pi^{ij}\partial_i f_1\partial_2 f_2 + \cdots , \qquad (4.19)$$

where $\cdots$ for $\Gamma = \tilde{\Gamma} = 0$ are given by the perturbation series, constructed in [6]. If one takes $\pi = const$, this series can be summed up, giving rise to the so-called Moyal product:

$$f_1 * f_2(x) = \exp\left[\pi^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\eta^j}\right] f_1(\xi)g(\eta)|_{\xi=\eta=x} \,. \tag{4.20}$$

4.5 The bulk algebra of Poisson diffeomorphisms is classical

Here we will not study the general symmetries of the C-model, but we will restrict ourselves to the diffeomorphisms that preserve the BV-action deformed by the Poisson bi-vector π, i.e.

$$\mathcal{L}_V\pi = 0\,. \tag{4.21}$$

Moreover we shall consider hamiltonian vector fields (on the simply-connected X all Poisson vector fields are hamiltonian) :

$$V_h^i = \pi^{ij}\partial_j h(Q) \tag{4.22}$$

One can check that such vector fields form the classical algebra:

$$[\mathcal{L}_{V_h}, \mathcal{L}_{V_g}] = \{h, g\}_\pi \,. \tag{4.23}$$

The currents that correspond to these classical diffeomorphisms are the 1-form components of the superfield $V_h^m(Q)Q_m^+$ and are equal to

$$J_{V_h} = V_h^m(q)p_m + \partial_k V_h^m p_m(0)q_{(1)}^k \,. \tag{4.24}$$

It is not trivial, one can check (Kontsevich technical lemma [6]) that the bulk algebra of currents coincides with the classical algebra (4.23).

Now we are in position to show that the algebra of the action of the currents on boundary observables is *not* classical, thus presenting an example of how the bubbling phenomenon works.

4.6 Action of the currents on the boundary observables in the C model

Consider the boundary observable $\mathcal{O}_f(x)$ that corresponds to the function $f(q)$ placed at point x. The current J_{V_h} is integrated along the arc with the endpoints, that we will denote as x_- and x_+.

If we consider π perturbatively, the U_{V_h} operation would be a series in powers of bi-vector π:

$$U_{V_h}(f)(x) = \left\langle \mathcal{O}_f(0)\int_{x_-}^{x_+} J_{V_h}\right\rangle_x \,. \tag{4.25}$$

The leading term is equal to the classical expression $\mathcal{L}_V f$, but there are other terms. One can explicitly compute them, using Konstevich diagrams, but instead we can use a shortcut.

We use the fact that the 1-observable that corresponds to the Poisson vector field is a sum of a Q_{BV}-exact term and a total derivative, namely:

$$(V_h^m Q_m^+)_{(1)} = dh + Q_{BV}(h)\,. \tag{4.26}$$

Thus, within correlator one has:

$$\int_{S(x,r)} (V_h^m Q_m^+)_{(1)} = \int_{S(x,r)} dh = \lim_{y\to x_+} h - \lim_{y\to x_-} h\,. \tag{4.27}$$

Now let us assume that $\pi = const$. One can show that in this case the limit coincides with the boundary value [8], and we get that

$$\left\langle \dots \int_{S(x,r)} J_{V_h} \mathcal{O}_f(x) \dots \right\rangle = \langle \dots \mathcal{O}_f(x)\mathcal{O}_h(x_+) \dots \rangle - \langle \dots \mathcal{O}_f(x)\mathcal{O}_h(x_-) \dots \rangle\,. \tag{4.28}$$

The r.h.s. of (4.28) can be computed with the help of the star-product.

Thus, from equation (4.28), we obtain the following result for U_{V_h}:

$$U_{V_h}(f) = h * f - f * h = [h \overset{*}{,} f]\,. \tag{4.29}$$

An obvious calculation shows that U_V commute via $*$-commutator rather than via Poisson commutator. The difference is just the manifestation of the disk bubbling phenomena, mentioned above.

In the case $\pi = const$ one can actually use this trick and compute all Kontsevich diagrams for an arbitrary vector field V, not necessarily Poisson. One finds:

$$U_V f(x) = \hat{A}^{-1}\left[\pi^{ij}\frac{\partial}{\partial \xi^i}\frac{\partial}{\partial \eta^j}\right] V^i(\xi)\partial_i f(\eta)|_{\xi=\eta=x}\,, \tag{4.30}$$

where $\hat{A}^{-1}(z) = \frac{e^z - e^{-z}}{z}$. This operation on functions has already appeared in [10].

5. Discussion and conclusions

In this paper, we have considered the realization of the target space diffeomorphisms in the type C topological string. Specifically we looked at the symmetries of closed and open string theories. In particular, we analyzed the algebra of charges which generate the infinitesimal transformations preserving closed string background π. The charges are given by the integrals of currents over little circles surrounding the bulk observables, and little arcs surrounding the boundary observables. We found that the charges, which generate infinitesimal transformations of the boundary observables, form an algebra, that is a deformation of the algebra of bulk charges, by the contribution of the bubbled disks.

The worldsheet perturbation technique developed in [6] can be applied to define infinitesimal transformations U_V in more general context, corresponding to general, not necessary Poisson vector fields V:

$$U_V f = V^i \partial_i f + \pi^{ij}\partial_i V^m \partial^2_{jm} f + \cdots\,. \tag{5.1}$$

They are a specific component of the L_∞-morphism of Kontsevich:

$$U_V f = \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{F}_1(V, \pi, \dots n \text{ times} \dots, \pi)[f]\,.$$

One can also establish that these operations make the $*$-product covariant in the sense that

$$U_V(f*g)-(U_Vf)*g-f*(U_Vg)=\mathcal{L}_V\pi\frac{\delta}{\delta\pi}(f*g) \tag{5.2}$$

and that they form the algebra

$$[U_{V_1},U_{V_2}]f-U_{[V_1,V_2]}f=[F_{BD}(V_1,V_2)\overset{*}{,}f]\,, \tag{5.3}$$

where $F_{BD}(V_1,V_2)=\sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{F}_0(V_1,V_2,\pi,\ldots n\text{ times},\ldots)$.

These properties can be formally established from the L_∞-morphism properties [11, 9, 12], but it is instructive to understand them from the world-sheet point of view. We showed that the deformation of the algebra is due to the phenomenon of disk "bubbling" which is purely field-theoretic effect.

This result can be applied to construct a "covariant" action of a noncommutative scalar field theory on $\mathbb{R}^n_\pi$ - the noncommutative space, whose algebra of functions is the $*$-product algebra. The field of this theory is an element ϕ of the algebra $\mathbb{R}^n_\pi$, and the action is given by:

$$\mathcal{S}=\int_{\mathbb{R}^n}g^{ij}[D_i,\phi]*[D_j,\phi]+V(\phi)\,, \tag{5.4}$$

where $V(\phi)$ is some polynomial function, say $(1/2)m^2\phi^2+\frac{\lambda}{4!}\phi*\phi*\phi*\phi$, and g^{ij} is a constant matrix. Then (5.2) implies that the correlation functions in such theory will be invariant under the transformations:

$$\begin{aligned} D_i &\mapsto D_i+U_VD_i \\ \pi &\mapsto \pi+\mathcal{L}_V\pi \\ \phi &\mapsto \phi+U_V\phi \end{aligned} \tag{5.5}$$

which could be used to define an improved stress-energy tensor (cf. [10]).

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